The question now is whether $\cos(2\pi/n)$ can be a root of a rational equation of lower degree. Let it be a root of an equation of degree k. The substitution of the exponential form for the cosine then shows that $e^{2i\pi/n}$ is a root of a rational equation of degree $2k$. But the equation of primitive nth roots of unity is not rationally reducible;[1] that is, $e^{2i\pi/n}$ is not a root of any rational equation of degree less than $\varphi(n)$. Therefore k is not less than $\frac{1}{2}\varphi(n)$. We have then the theorem: *If n is an integer greater than 2, $\cos(2\pi/n)$ is a root of an irreducible equation of degree $\frac{1}{2}\varphi(n)$.*

The simpler cases are:

$\varphi(n) = 2$ when $n = 3, 4, 6$; $\cos(2\pi/n)$ rational, as also for $n = 1, 2$.
$\varphi(n) = 4$ when $n = 5, 8, 10, 12$; $\cos(2\pi/n)$ a quadratic surd.
$\varphi(n) = 6$ when $n = 7, 9, 14, 18$; $\cos(2\pi/n)$ a cubic surd.
$\varphi(n) = 8$ when $n = 15, 16, 20, 24, 30$; $\cos(2\pi/n)$ a quartic surd.

DISCUSSIONS.

Professor Hathaway obtains an integral reduction formula which includes as special cases the formulas usually given in works on the integral calculus.[2] In a note following the paper it is shown how the formula may be regarded as a transform of one of these special cases. This, of course, does not prevent it from being also a generalization.

A General Type of Reduction Formula.

By A. S. Hathaway, Rose Polytechnic Institute.

Let X, Y, Z be functions of a single variable, such that

$$AX^2 + BY^2 + CZ^2 = 0, \tag{1}$$

A, B, C being constants. By differentiation,

$$AXdX + BYdY + CZdZ = 0; \tag{2}$$

and from (1) and (2)

$$\frac{YdZ - ZdY}{AX} = \frac{ZdX - XdZ}{BY} = \frac{XdY - YdX}{CZ} = dT \text{ (for brevity).} \tag{3}$$

The general integral considered is

$$\beta(L, M, N) = \int X^L Y^M Z^N dT, \text{ where } L + M + N + 1 = 0. \tag{4}$$

The integral is homogeneous of order zero. It is not altered by substituting for X, Y, Z any variable common multiples of them, VX, VY, VZ. These also satisfy (1) and (3).

Further the integral is not altered by permuting X, Y, Z, concurrently with A, B, C and L, M, N; except that the sign is changed when the permutation is not cyclic (from the definition of dT). This means that any formula of expansion of the integral in powers of two bases is equivalent to six different formulas

[1] Dedekind's proof for the general case (n composite) is given in H. Weber, *Lehrbuch der Algebra*, volume 1 (Braunschweig, 1898), p. 596, or *Traité d'Algèbre Supérieure* (French translation by J. Griess, Paris, 1898), p. 636. For proof by Arndt, see P. Bachmann, *Die Lehre von der Kreistheilung* (Leipzig, 1872), p. 38.

[2] W. A. Granville, *Elements of the Differential and Integral Calculus*, Boston, 1911, pp. 350-360.

obtained by the six different permutations of concurrent parts. We propose to establish such a formula, and show the great variety of its applications.

We have $d(X^{L+1}Y^{M-1}Z^{N+1}) = X^LY^{M-2}Z^N[(L+1)YZdX + (M-1)ZXdY + (N+1)XYdZ] = X^LY^{M-2}Z^N[(M-1)CZ^2 - (N+1)BY^2]dT.$

Integrating and solving for $\beta(L, M, N)$,

$$\beta(L, M, N) = \frac{X^{L+1}Y^{M-1}}{B(-N-1)Z^{-N-1}} + \frac{-C(M-1)}{B(-N-1)}\beta(L, M-2, N+2). \quad (5)$$

The reduced integral is of the same form, since one exponent is increased as much as the other is decreased; and (5) applies to it, with proper change of exponents, M to $M-2$, N to $N+2$. This can be continued in general indefinitely. The resulting formula[1] is:

$$\begin{aligned}\int X^LY^MZ^NdT &= \frac{X^{L+1}Y^{M-1}}{B(-N-1)Z^{-N-1}} + \frac{-C(M-1)X^{L+1}Y^{M-3}}{B^2(-N-1)(-N-3)Z^{-N-3}} + \cdots \\ &+ \frac{(-C)^{k-1}(M-1)(M-3)\cdots(M-2k+3)X^{L+1}Y^{M-2k+1}}{B^k(-N-1)(-N-3)\cdots(-N-2k+1)Z^{-N-2k+1}} \\ &+ \frac{(-C)^k(M-1)\cdots(M-2k+1)}{B^k(-N-1)\cdots(-N-2k+1)}\int X^LY^{M-2k}Z^{N+2k}dT. \quad (6)\end{aligned}$$

If M were an odd positive integer, we would have an algebraic integral of k terms, where $k = (M+1)/2$, since a factor zero appears in the coefficient of the reduced integral. We could not carry the reduction so far, however, if at the same time N were an odd negative integer not numerically larger than M, as a zero factor would appear first in the denominator, and the series must be stopped before that occurs. The effect of a possible zero factor in stopping reduction may be thus stated: *an odd exponent cannot be reduced so that its sign changes.* If positive, it must stop at 1; if negative, at -1. We have: *the integral $\beta(L, M, N)$ is algebraic in X, Y, Z when (and only when) one exponent is an odd*

[1] Equation (6) may be written:

$$\beta(L, M, N) = X^{L+1}\left\{\frac{-C\cdot M-1\cdot Y}{B\cdot -N-1\cdot Z}k\right\} + \left\{\frac{-C\cdot M-1}{B\cdot -N-1}k\right\}\beta(L, M-2k, N+2k)$$

The bracketed terms are "indices" for corresponding sums and products, and k is the number of integrated terms.

By permuting, Y and Z become *any two of the variables*, M and N, their exponents, B and C, their coefficients in (1), *with the factor -1 before that coefficient which follows the other in the cyclic order $ABCA$.*

In this notation, the illustrative example given later is,

$$\beta(4, 6, -11) = X^5\left\{\frac{3\cdot 5\cdot Y}{2\cdot 10\cdot Z}3\right\} + \left\{\frac{3\cdot 5}{2\cdot 10}3\right\}\beta(4, 0, -5).$$

$$\beta(4, 0, -5) = Y\left\{\frac{-3\cdot 3\cdot X}{-7\cdot 4\cdot Z}2\right\} + \left\{\frac{-3\cdot 3}{-7\cdot 4}2\right\}\beta(0, 0, -1).$$

A complete reduction may therefore be written in this notation, given variables, exponents, and coefficients. Preferably reduce the two *numerically greatest* exponents, one of which is positive, the other negative, by (4).

positive integer, the other two being any numbers subject to (4), *rational or irrational, that are not both negative odd integers.*

When one exponent is a positive odd integer and the other two negative odd integers, the integration may be made with a logarithmic term. For the exponents reduce to $1, -1, -1$; and

$$\int \frac{XdT}{YZ} = \frac{1}{A} \int \left(\frac{dZ}{Z} - \frac{dY}{Y} \right) = \frac{1}{A} \log \frac{Z}{Y}.$$

Hence $\beta(L, M, N)$ *can always be integrated when one exponent is a positive odd integer.*

The integration is possible if one exponent be an odd negative integer and the other exponents rational numbers. For integral exponents (excluding an odd positive one, already considered), by (4), two must be even and one odd negative, so that the exponents reduce to $-1, 0, 0$. In this case, $\int dT/X = \int (YdZ - ZdY)/(AX^2) = - \int (YdZ - ZdY)/(BY^2 + CZ^2) = \int dx/(Bx^2 + C)$, where $x = Y/Z$ (an anti-tangent or logarithm as the signs of B, C are alike or not).

With one odd negative and two fractional exponents, the exponents reduce to $-1, m/n, -m/n$, by (4), and we have the preceding form multiplied by $x^{m/n}$ $(x = Y/Z)$, which integrates as a rational fraction by the substitution $x = z^n$.

It seems probable that $\beta(L, M, N)$ is not integrable in elementary functions when no exponent is an odd integer. By (4), there must be an odd exponent when all are integers.

In reducing by (6), it is first necessary to determine bases, coefficients, and exponents. Powers of X, Y, Z must be factors of the given differential to be integrated, their relation (1) being found by inspection. Then dT is computed by (3), and the quotient of the differential by dT must be, to a constant factor, $X^L Y^M Z^N$, where $L + M + N = -1$. If there are only two functions X, Y, with a relation $AX^2 + BY^2 + C = 0$, then $Z = 1$, $dT = dX/BY$, and the relation (4) is no restriction, but only a determination of the exponent of $Z = 1$, required in (6).

Of the six possible permutations of formula (6), only two can be used to reduce exponents to their smallest values (between 1 and -1). Namely, there must be a positive exponent in place of M, and a negative one in place of N (only two, since by (4) two exponents are positive and one negative, or vice versa, or all are negative and reduced). Generally either pair of exponents may be used to reduce its numerically smallest exponent (leaving the third unchanged), and in the reduced integral there is only one pair for further reduction, and this pair leaves the exponent just reduced unchanged.

For example, find $\int x^4(3 + 4x^2)^{5/2}(2 + 5x^2)^{-6}dx$. The variables are $X = x$, $Y = (3 + 4x^2)^{1/2}$, $Z = (2 + 5x^2)^{1/2}$, with $7X^2 + 2Y^2 - 3Z^2 = 0$. $dT = dx/(YZ)$, and the given integral is therefore $\int X^4 Y^6 Z^{-11} dT = \beta(4, 6, -11)$, since $4 + 6 - 11 + 1 = 0$.

To apply (6) we have a choice between the pairs of exponents $(4, -11)$ and

(6, − 11). Taking the latter, we have

$$\beta(4, 6, -11) = X^5\left[\frac{Y^5}{2\cdot 10\cdot Z^{10}} + \frac{3\cdot 5Y^3}{2^2\cdot 10\cdot 8Z^8} + \frac{3^2\cdot 5\cdot 3Y}{2^3\cdot 10\cdot 8\cdot 6Z^6}\right] + \frac{3^3\cdot 5\cdot 3\cdot 1}{2^3\cdot 10\cdot 8\cdot 6}\beta(4, 0, -5).$$

For next reduction we have only the pair (4, − 5).

$$\beta(4, 0, -5) = Y\left[\frac{X^3}{-7\cdot 4Z^4} + \frac{-3\cdot 3X}{7^2\cdot 4\cdot 2Z^2}\right] + \frac{3^2\cdot 3\cdot 1}{7^2\cdot 4\cdot 2}\beta(0, 0, -1),$$

$$\beta(0, 0, -1) = \int dT/Z = \int\frac{XdY - YdX}{-3Z^2} = \int\frac{XdY - YdX}{7X^2 + 2Y^2} = \frac{1}{\sqrt{14}}\tan^{-1}\frac{\sqrt{2}Y}{\sqrt{7}X}.$$

The solution is complete on substituting values found.

Note the following particular form:

$$\tfrac{1}{2}\int(a + bx)^{(L-1)/2}(a' + b'x)^{(M-1)/2}(a'' + b''x)^{(N-1)/2}dx = \beta(L, M, N), \quad (L + M + N + 1 = 0).$$

$$X^2 = a + bx, \qquad Y^2 = a' + b'x, \qquad Z^2 = a'' + b''x,$$

$$(a'b'' - a''b')X^2 + (a''b - ab'')Y^2 + (ab' - a'b)Z^2 = 0.$$

$$dT = dx/(2XYZ).$$

Of other forms, note:

$$\int\sin^L x\cos^M x\,dx; \qquad X = \sin x, \qquad Y = \cos x, \qquad Z = 1,$$

$$X^2 + Y^2 - Z^2 = 0, \qquad dT = dx.$$

$$\int\tan^L x\sec^{M+1} x\,dx; \qquad X = \tan x, \qquad Y = \sec x, \qquad Z = 1.$$

$$-X^2 + Y^2 - Z^2 = 0, \qquad dT = \sec x\,dx.$$

In each of these the exponent of Z is $N = -L - M - 1$.

Note by the Editor.

In the application of the reduction formula, the problem is to recognize the four variables X, Y, Z, and T. It therefore suggests itself that we should look a little into the meaning of these variables in the integral. The equations (1) and (3) show that only two of the four are really independent, so that the formula in question ought to be transformable into a canonical form containing only one arbitrary function. But actually the situation is simpler still, on account of the homogeneity to which the author has alluded, which makes the ratios of X, Y, Z the only functions of importance.

In dealing with real quantities, we must be able to throw (1) into some such form as $(Z/c)^2 = (X/a)^2 + (Y/b)^2$. We may then use two variables Z and θ, so that $X = (a/c)Z\cos\theta$, and $Y = (b/c)Z\sin\theta$. (Geometrically, we are dealing with points (X, Y, Z) on a quadric cone, and θ is the eccentric angle in a principal elliptic section.) It follows that $dT = c^2(XdY - YdX)/Z = abZd\theta$.

Now it must be supposed that we integrate along some path lying on the cone and given by an equation $Z = F(\theta)$. But the homogeneity principle prepares us to see Z disappear from the integral, as indeed it does, the result of the transformation being $\int X^L Y^M Z^{-L-M-1} dT = a^{L+1} b^{M+1} c^{-L-M} \int \cos^L \theta \sin^M \theta \, d\theta$.

Thus every integral of the type in question is transformable into this one particular form. In the author's first example, the required transformation is

$$\cos \theta = \sqrt{\frac{7}{3}} \cdot x(2 + 5x^2)^{-1/2}.$$

It may be added that, if we possess the general reduction formula developed above, nothing is gained by this transformation from the point of view of integration. In fact the reduction formula is exactly the same before and after transformation.

RECENT PUBLICATIONS.

REVIEWS.

The Principles of Geometry. By H. F. Baker. Volume 1, Cambridge University Press, 1922. 182 pp. Price 12 shillings.

That prince of teachers, the late Jules Tannery, once wrote in the introduction to a book by one of his former pupils: "Un petit livre est rassurant."

When a book is not only short but well printed, with wide margins and in a flowing style of the King's English, the reassuring impression is much strengthened. But the reader who takes up the work before us under these pleasant impressions, with the idea that in spite of a lack of any special preparation in the way of familiarity with the subject matter and point of view, he is going in a few hours to reach the real substance, this reader will have a very prompt chance to guess again.

It will take a much greater convulsion than the late World War to stop the output of books on Projective Geometry, especially from English writers. Yet so far, in England, they have clung to the tradition of Cremona and Stephen Smith, taking the metrical definition of cross ratios as fundamental, regardless of the fact that Klein proved half a century ago that the cross ratio can be reached by purely descriptive processes, and that Continental and American writers have been doing this in their texts for a generation. Sooner or later some Englishman was bound to fall into line, and it is a subject for satisfaction that the first should be such a distinguished scholar as Baker. The reason for the delay is doubtless this, that English boys are drilled in a subject of which Americans have scarcely heard the infelicitous name, Geometrical Conics. The metrico-projective treatment flows from this in the most natural manner. Moreover the present writer does not say that he is actually writing a book on projective geometry, and he starts his second chapter with a discussion of the descriptive properties of a limited region. But as soon as he has strengthened this to the point where it can stand on its legs, he adjoins to the universe of discourse ideal,